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NUMERICAL CALCULATION OF RELATIVISTIC MULTIPLE-CAVITY SYSTEMS

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In the calculation of various electrophysical devices which use relativistic electron beams, it is necessary to consider the motion of a beam of charged particles in an external electromagnetic field and the self-fields (irrotational and solenoidal) of the beam. A multiple-cavity klystron is an example of such a device, in which the interaction between the relativistic electron beam and the radiation field in the cavities is significant. The numerical treatment of such processes is based on Maxwell's equations.

In the present paper we describe the numerical algorithms and their computer program implementation in the framework of the package of applied programs ÉRANS [1] for the calculation of a relativistic beam of charged particles moving in extended multiple-cavity systems. The problem is split up into the following subproblems: 1) the calculation of the input cavity into which an ungrouped flux of charged particles enters (oscillations in the cavity are excited and maintained by an external source, such as a current loop); 2) the calculation of the flux in the drift tubes; 3) the calculation of the flux in the relay and output cavities; 4) the joining of the solutions of the first three problems.

The problem is assumed to be axisymmetric and is treated in terms of the cylindrical coordinates r, z, θ, where the motion of the beam is mostly along the axis of symmetry z.

Economy of calculation is the basic criterion used in choosing the numerical algorithms. In carrying through the calculations for different parts of the system, the most significant factors affecting the flux of charged particles for that part of the system are taken into account. Inside the cavities the solenoidal fields $\mathbf{E}^S$, $\mathbf{H}$ are taken into account, where the nonzero components of these fields are E_r^S, E_z^S, and H_θ (the so-called E-field). In the drift tubes we take into account the azimuthal component of the self-magnetic field of the particles beam. External electric and magnetic fields act on the beam over the entire system, as does the irrotational field of the beam. The first five harmonics of the vector potential (see below) are used to calculate the solenoidal fields.

The algorithms allow one to follow transient processes in separate parts of the system; however, in the present paper we will be concerned mainly with steady-state, periodic processes. Our approach is illustrated on a problem of practical interest.

We consider separately the algorithms for the solution of the above subproblems. The discussion is ordered in a convenient way for the description of the algorithms. The problem of calculating the flux of charged particles in a resonant cavity reduces to finding the solution of the complete set of Maxwell's equations

$$\operatorname{div} \mathbf{E} = \rho/\varepsilon_0; \tag{1}$$

$$\operatorname{rot} \mathbf{E} = -\mu_0 \partial \mathbf{H}/\partial t; \tag{2}$$

$$\operatorname{div} \mathbf{H} = 0; \tag{3}$$

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$$\operatorname{rot} \mathbf{H} = \mathbf{j} + \varepsilon_0 \partial \mathbf{E}/\partial t, \tag{4}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, ρ is the space charge density, $\mathbf{j}$ is the current density, ε_0 is the permittivity of free space, μ_0 is the magnetic permeability of free space, ($\varepsilon_0\mu_0 = c^{-2}$, c is the speed of light), t is the time.

Maxwell's equations (1)-(4) must be supplemented by the equation of motion of the charged particles (with charge e)

$$d\mathbf{p}/dt = e(\mathbf{E} + \mu_0[\mathbf{vH}]). \tag{5}$$

Here $\mathbf{p} = \gamma m\mathbf{v}$ is the momentum of the particle, m is its rest mass, $\mathbf{v}$ is the velocity, $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor, $v = |\mathbf{v}|$.

Solution of the system (1)-(4) reduces [2] to determining the solenoidal $\mathbf{E}^S$, $\mathbf{H}$ and irrotational $\mathbf{E}^I$ fields: $\mathbf{E} = \mathbf{E}^S + \mathbf{E}^I$, $\operatorname{div} \mathbf{E}^S = 0$, $\operatorname{rot} \mathbf{E}^I = 0$. The solenoidal fields are calculated using equations

$$\mathbf{E}^S = -\sum_{k=1}^{\infty} \frac{dq_k(t)}{dt} \mathbf{A}_k^S(\mathbf{r}), \quad \mathbf{H} = \frac{1}{\mu_0} \sum_{k=1}^{\infty} q_k(t) \operatorname{rot} \mathbf{A}_k^S(\mathbf{r}), \tag{6}$$

where $\mathbf{r}$ is the position vector of the observation point and the $\mathbf{A}_k^S(\mathbf{r})$ are the vector potentials satisfying

$$\Delta \mathbf{A}_k^S + \eta_k \mathbf{A}_k^S = 0, \quad (\mathbf{A}_k^S)_{tg}|_\Gamma = 0 \tag{7}$$

(Δ is the Laplacian, η_k is the eigenvalue, Γ is the boundary of the region under consideration, $(\mathbf{A}_k{}^S)_{tg}$ is the tangential component of the vector), and the coefficients $q_k(t)$ are determined from the solution of the equation

$$\frac{d^2 q_k}{dt^2} + \frac{\omega_k}{Q_k} \frac{dq_k}{dt} + \omega_k^2 q_k = \int_V \mathbf{j}(\mathbf{r}, t) \mathbf{A}_k^S(\mathbf{r})\, dV, \tag{8}$$

in which Q_k is the Q-factor of the cavity at frequency $c\sqrt{\eta_k}$.

The irrotational field is found from the solution of Poisson's equation

$$\Delta\varphi = -\rho/\varepsilon_0, \quad \varphi|_{\Gamma_1} = 0, \quad \left.\frac{\partial\varphi}{\partial \mathbf{n}}\right|_{\Gamma_2} = 0. \tag{9}$$

Here φ is defined such that $\mathbf{E}^I = -\operatorname{grad} \varphi$; $\Gamma_1 \cup \Gamma_2 = \Gamma$.

The numerical algorithms for these problems are similar to those considered in [1] and include the following basic steps: solution of the eigenvalue problem (7); calculation of the solenoidal and irrotational fields; integration of the equations of motion; calculation of the current density and space charge density.

An unsteady process in the cavity is represented as follows. The time interval $[T_I, T_F]$ under consideration is split up into N_T time steps $\Delta T_i = T_i - T_{i-1}$ ($i = 1, 2, \ldots, N_T$; $T_I = T_0 < T_1 < \ldots < T_{N_T} = T_F$). The eigenfunctions of the vector potential are computed and stored in the database of the program package ÉRANS. The equations of motion are integrated to $t = T_1$, using the electric and magnetic fields calculated at time $t = T_I$. The solution of the equations of motion is used to calculate the space charge and current density distributions. Using these distributions, the solenoidal and irrotational fields are recalculated at $t = T_1$ and the fields are assumed to be independent of time over the time interval ΔT_2. The charged particles are then propagated to the time T_2 and the whole process is repeated until the time $t = T_F$ is reached.

The program ÉDIP [3] was used to calculate the eigenfunctions of the vector potential. The flux of charged particles is modeled by the "large" particle method. The equations of motion are solved numerically on the time interval $[T_{i-1}, T_i]$ using the scheme described in [1] with a stepsize $\tau^i \leq \Delta T_i$. The solenoidal fields are calculated using [6], in which not more than the first five terms of the series are retained in the calculation.

The right-hand side of (8) is evaluated by using the values of the integrand at the nodes of the computational network [1]. The following approximation formula is used to solve (8) in the interval $t \in [T_{i-1}, T_i]$:

$$x_k^i = x_k^{i-1} + \left(F_k^i - \frac{\omega_k}{Q_k}\frac{x_k^i + x_k^{i-1}}{2} - \omega_k^2 \frac{q_k^i + q_k^{i-1}}{2}\right)\Delta T_i,$$

$$q_k^i = q_k^{i-1} + \frac{x_k^i + x_k^{i-1}}{2}\Delta T_i,$$

where $F_k{}^i$ are the values of the right-hand side of (8) at the nodes and $x_k{}^i = (dq_k/dt)^i$. The potential fields [given by (9)] are determined by means of a difference algorithm on a rectangular nonuniform network. The system of difference equations is solved by iteration [4].

The calculation of the input cavity has the following features. Oscillations in the cavity are excited and maintained by a current $I(t) = I_0 \cos \omega t$ (I_0 = const, ω is the frequency) passing through a loop, which is approximated by a rectangle R: $z_i{}^R \le z \le z_f{}^R$, $r_i{}^R \le r \le r_f{}^R$. The amplitudes $q_k(t)$ of the forced oscillations in the cavity are found from (8) with the right-hand side given by $\mu_0 I(t) s_R \left(s_R = \int_{z_I^R}^{z_F^R}\int_{r_I^R}^{r_F^R} H_{\theta_k}\, dr dz, \right.$ H_{θ_k} is the θ component of the magnetic field induced in the cavity). The integral is evaluated numerically using second-order formulas. In the case considered here, (8) has the unique periodic solution $q(t) = \mu_0 I_0 Q_F \omega_F{}^{-2}\, s_R \sin \omega t$ at $\omega = \omega_k$.

In addition to these forced E-oscillations, the self-electric and magnetic fields and the external field also act on the charged particles in the input cavity. The effect of the self-magnetic field is taken into account for weak grouping of the charged particles by recalculating the r component of the irrotational electric field $E_r{}^I$ [5]: $E_{rN}{}^I = E_r{}^I/\gamma^2$ ($E_{rN}{}^I$ is the "new" value of the field). The remaining calculations of the input cavity are carried out using algorithms similar to those used in the calculation of the relay cavity.

The flux of charged particles in the drift tubes is calculated assuming that the radiation fields in the tubes are small. The effects of the space charge and the self-magnetic field of the beam are taken into account, as well as the external fields. The irrotational electric fields in the drift tubes are calculated by difference methods. To save execution time, the difference network is chosen to be uniform in this case and the system of difference equations constructed on the network is solved directly by cyclic reduction. A modification of this method was implemented in which the potential can be calculated using an arbitrary (not only $2^k + 1$, where k is an integer) number of nodes along each of the coordinate axes [6].

The effect of the azimuthal component of the self-magnetic field on the charged particles moving in the tube was taken into account. This field was calculated from the formula [7] $H_\theta = I/2\pi R$, where I is the conduction current through a cross section S of radius R (the displacement current $I_d = \varepsilon_0 \int_S \frac{\partial E}{\partial t}\, dS$ is taken to be small in view of the assumed nature of the fields in the tube). For weak grouping of the particle beam, the effect of the self-magnetic field can be taken into account using the algorithms described above.

The solutions of the subproblems are joined together to form the solution of the original problem in the following way. The computational region is divided up by the planes z = const into a set of subregions, each of which is either the input cavity, a drift tube, or the relay (output) cavity. The coordinates, velocities, and charges of the "large" particles leaving any of the subregions during each time step are stored in memory. This information is then read from memory and used as the initial conditions to continue the calculations into the next subregion. In this way, the solutions are joined from one subregion to the next. The boundary condition for the electric potential φ on the planes bounding a given subregion is $\partial\varphi/\partial z|_{z=\text{const}} = 0$, which is satisfied accurately in the first few subregions of the multiple-cavity system. In the succeeding subregions it is necessary to take the potential calculated in the preceding subregion as boundary conditions.

This approach is the analog of a single iteration of the alternating method of Schwartz [8]. The numerical simulations show that the beam characteristics in the second iteration differ from those of the first iteration by a fraction of a percent. This indicates that

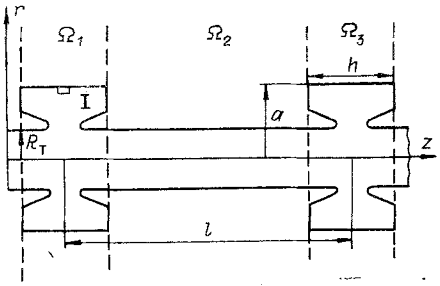

Fig. 1

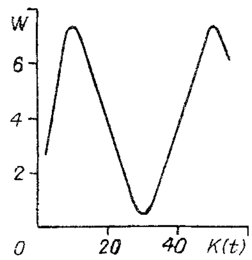

Fig. 2

a single iteration is sufficient to obtain a solution of the problem with acceptable accuracy. In the calculations in the relay cavity the convergence of the unsteady process to a periodic one was speeded up by using a solution obtained from the one-dimensional analytical theory [9] as the initial approximation.

The numerical algorithms discussed here were run on the BÉSM-6 computer within the package of applied programs ÉRANS [1], which consists of a library of program modules, a database, and the programming languages. The modules of the package were coded in FORTRAN, ALGOL-GDR, and MADLEN and they are maintained in libraries on magnetic tape or disk. The database of the package, stored on magnetic drums, tape, or disk, contains information needed to interrupt and restart the calculation, to join the subregions, and also contains the eigenvalues and eigenfunctions of the cavities. The input data for the problem and information on derived characteristics, such as the solenoidal fields and the radiative energy as functions of time, the field patterns at given time steps, the coordinates, velocities, and charges of the "large" particles, and so on, are described in [10, 11]. The results are presented in the form of tables of numbers or plots.

This package of programs was used to carry out the calculation for one element of a relativistic multiple-cavity klystron used in UHF energetics of linear accelerators (Fig. 1). The element consists of two cavities (an input cavity Ω_1 and a relay cavity Ω_3) and a drift tube Ω_2 between them. A tubular beam with a uniform current density enters the first cavity, whose parameters are $a = 1.3 \cdot 10^{-2}$ m, $h = 1.5 \cdot 10^{-2}$ m. The beam moves in a magnetic field directed along the z axis ($H_r = 0.12 \cdot 10^6$ A/m in the first cavity and $H_T = 0.24 \cdot 10^6$ A/m in the other subregions). The beam parameters on entry into the computational region were chosen as: outer radius $R_F = 4 \cdot 10^{-3}$ m, inner radius $R_I = 10^{-3}$ m, $I = 50$ A, $\gamma = 1.4$. The beam obtains an initial velocity modulation inside the input cavity. Inside the drift tube (of radius $R_T = 5 \cdot 10^{-3}$ m, length $l = 9 \cdot 10^{-2}$ m) the beam is grouped with respect to space charge density. The grouped beam then excites oscillations in the relay cavity, which has the same geometrical dimensions as the input cavity. The wavelength of the fundamental mode of oscillation is $\lambda_1 \approx 4 \cdot 10^{-2}$ m. The Q-factors of the input and relay cavities were calculated using the linear analytical theory in the one-dimensional approximation [9] with the charging of the cavities by the beam taken into account.

The following computational parameters were used in the calculations: in the input and relay cavities the number of nodes of the difference network was $N_n \simeq 1000$ and the number of large particles was $N_p \simeq 200$; in the drift tube $N_n \simeq 3000$, $N_p \simeq 1200$.

The period T_c of the oscillations in the input cavity was divided into 40 time steps. The time step was constant in all of the calculations. The value of the function $\sigma(t) =$

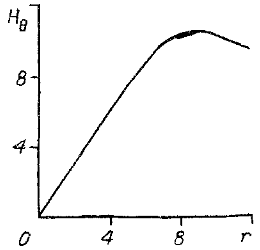

Fig. 3

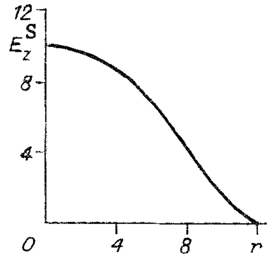

Fig. 4

$\int_0^h E_z^S(0, z, t)\,dz$ was computed at each time step. The periodicity of the solution in the cavities was controlled using the quantities $\varepsilon_i^t = \frac{t_{i+1}^C - t_i^C}{t_i^C}$, $\varepsilon_i^m = \frac{|\sigma_{i+1}^m - \sigma_i^m|}{\sigma_i^m}$, where t_i^C (i = 1, 2, ...) are the zeroes of σ(t), and $\sigma_i^m = \max\limits_{t_i^C < t < t_{i+1}^C} |\sigma(t)|$. For both cavities the quantities ε_i^t, ε_i^m decreased to fractions of a percent after 80 time steps, which took about 0.5 h of execution time on the BÉSM-6. The periodicity of the process in the drift tube was established after 240 time steps (2.5 h of execution time) and was verified by checking that the coordinates and velocities of "large" particles at times t and t + T_C were equal, to within a fraction of a percent, for each cross section z = const.

As an example of the results obtained from the numerical calculations, the characteristics of the electromagnetic field in the relay cavity Ω_3 are shown in Figs. 2-4. Figure 2 shows the time dependence of the energy of the electromagnetic field $W = \frac{1}{2}\varepsilon_0 \int_V (E^S)^2 dV + \frac{1}{2}\mu_0 \int_V H_\theta^2 dV$, the function $H_\theta(r)$ for z = const at the center of the cavity is shown in Fig. 3 for a fixed value of the time, and Fig. 4 shows the function $E_z^S(r)$ for the same conditions. All quantities plotted in the figures are dimensionless: $r = r(m)/r_0$, $W = W(J)/W_0$, $H_\theta = H_\theta(V/m)/H_0$, $E_z^S = E_z^S(V/m)/E_0$, where $r_0 = 10^{-3}$ m, $W_0 = 10^{-3}$ J, $H_0 = 1.34 \cdot 10^2$ V/m, $E_0 = 10^5$ V/m. In Fig. 2, K_t is the number of the time step, where $K_t = K_t' - 100$ (K_t' is the number of the time step counted from the time of entry of the particles into Ω_3).

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